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# Vulnerability Analysis of the Financial Network

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Since the financial crisis in 2007-2008, the vulnerability of a financial system has become a major concern in financial engineering. In this paper, we analyze the vulnerability of a financial network based on the linear optimization model introduced by Eisenberg and Noe (2001), where the right hand side of the constraints is subject to market shock and only limited information regarding the liability matrix is exposed. We develop a new extended sensitivity analysis to characterize the conditions under which a bank is solvent, default or bankrupted, and estimate the probability of insolvency and the probability of bankruptcy under mild conditions on the market shock and the network structure. Particularly, we show that while an increment in the total asset may not able to improve the stability of the financial system, a larger asset inequality in the system will reduce its stability. Moreover, under certain assumption on the market shock and the network structure, we show that the least stable network can be attained at some network with a monopoly node, which also has the highest probability of insolvency. The probability of bankruptcy in the network when all the nodes receive shocks is estimated. We also estimate the impact of bankruptcy of the monopoly node in a well-balanced network and explore the domino effect of bankruptcy when the network has a tridiagonal structure. Numerical experiments are presented to verify the theoretical conclusions.

*Key words*: Financial network, Systemic risk, Linear optimization, Sensitivity analysis, (In)feasibility analysis, Stochastic optimization, Probability.

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## 1. Introduction

A typical financial network comprises multiple financial institutions interacting with each other through borrowing and lending or interconnecting indirectly through the market by holding similar shares or portfolios. The presence of tight linkages among the financial institutions has various consequences in the global financial market. On the one hand, it influences asset prices by acquiring and processing the related information more efficiently, and as a result, large numbers of transactions can be proceeded smoothly without any interruptions and the trading performance is improved. On the other hand, whenever some institution bankrupts in the system, it may lead to a catastrophic disaster by spreading this failure quickly over the entire system. This is usually referred as the so-called systemic risk. As evidence we recall the 2008 financial crisis in US that have triggered not only the entire U.S. financial industries but also several international financial markets around the world (Murphy 2008). Another example is the European sovereignty debt crisis that causes the European financial business to face serious loss of confidence (Lane 2012).

The catastrophic consequences from these widespread phenomena have prompted extensive study on the assessment of the systemic risk and the contagion effect in a financial network. In a seminal paper, Eisenberg and Noe (2001) introduce the clearing payment system considering bankruptcy law to assess the systemic risk in inter-banking networks. According to Eisenberg and Noe (2001), a financial institution in the network is said to bankrupt if it cannot payback any of its liabilities, default if it can payback only a fraction of its liabilities and be solvent when it can fulfill its liabilities. They show that the failure of a single institution can transmit to other financial institutions, leading to a contagion risk. Particularly, they propose various linear and nonlinear optimization models to measure the systemic risk in a financial network, and these optimization models variate in objective functions but do share the same set of constraints and the same unique optimal solution. They also propose an algorithm to compute the clearing payment vector by tracking the sequence of default.

In a series of papers, Elsinger et al. (2005, 2006), Elsinger (2009), Elsinger et al. (2013) study the financial stability of a banking system considering the cascading impact of failures over the entire network. In their model, the joint impact of two major sources of risk, the correlated exposure and domino effects, is considered. Specifically, Elsinger et al. (2005, 2006) estimate the systemic risk in the financial network in UK and Austrian based on data from banks in these two countries. Since the data from banks usually reveals only partial information regarding the interbank liabilities and the assets are subject to market fluctuation, Elsinger et al. (2005, 2006) suggest to first compute the liability matrix by solving some entropy optimization problem based on the so-called Kullback-Leibler (KL) divergence, and then use stochastic optimization and scenario generation to estimate the worst-case scenario for the underlying linear optimization model.

There exist several works focusing on the impact of interbank liability structure on the risk exposure. In their pioneering work, Allen and Gale (2000) first establish the connection between the specific pattern of interbank lending and the extend of contagion in a financial system. Gia and Kapadia (2010), Gia et al. (2011) discuss how contagion spreads in a random network, and analyze the knock-on effects of distress. Battiston et al. (2012a,b) study the effect of credit risk diversification and network density on systemic risk. Liu and Staum (2010) apply the standard sensitivity analysis in Eisenberg-Noe's LP model to estimate the impact of the market shock to a single financial institution. Acemoglu et al. (2013) study the stability of financial networks depending on the network structure and magnitude of negative shock to a single financial institution in the network. Glasserman and Young (2015) study the contagion effects via network spillovers under certain assumptions on the shock distribution and network structure. The survey Elsinger et al. (2013) summarizes various concepts and network optimization models and discuss their findings and applications in systemic risk analysis up to that time. For more details in this direction, we refer to Elsinger et al. (2013) and the references therein. Very recently, Chen et al. (2016) explore the optimality conditions in Eisenberg-Noe's linear optimization model to design a partition algorithm that can separate the default institutions and the solvent ones in the network. They also use sensitivity analysis to estimate the impact of both market and liquidity channel in risk transmission.

In this paper we focus on the vulnerability analysis of the financial network based on the linear optimization (LO) model introduced in Eisenberg and Noe (2001). Our work is inspired by a key observation, as pointed out in the survey Elsinger et al. (2013), that most existing works have underestimated the systemic risk in the financial system. In Elsinger et al. (2013), the authors further speculate that the incomplete information on the financial network may be one major reason for the underestimation of the risk. We notice that another possible reason for the underestimation of the risk in a financial system is the restrictive small shock assumption widely used in the existing literature. To see this, let us take a closer look at the reference Liu and Staum (2010) where the authors estimate the contagious risk under two assumptions: One assumption is that the complete information of a financial network is available and another assumption is that the market shock will not change the set of default banks and the set of solvent banks. As shown in Liu and Staum (2010), under these two assumptions, the contagious risk in the network can be estimated via the solution to the dual problem of the LP problem in Eisenberg and Noe (2001). Note that the assumption that the sets of default banks and solvent banks remain invariant implicitly implies that the market shock is insignificant or small, which is very different from what happened during the financial crisis in 2007-2008 when the large market shock had led to the bankruptcy of financial institution such as Lehman Brothers. Clearly, the small shock assumption cannot be used in the analysis of bankruptcy in a financial network. A challenge here is how to assess the systemic risk of a financial network when only limited and incomplete information regarding the financial network is available and the market shock is significant.

The primal goal of this work is to address the above challenge via developing a new theoretical framework to analyze the vulnerability of a financial network under mild assumptions on market shocks. To start, we mention that based on the LO model in Eisenberg and Noe (2001), the bankruptcy in a financial network corresponds to the infeasibility of the model itself. For a given linear optimization problem in which all the data are available, we can refer to the well-known Farkas lemma to detect its feasibility(Vanderbei 2000). Unfortunately, for the LO model in Eisenberg and Noe (2001), as observed in Elsinger et al. (2013), Upper and Worms (2004), only limited information on the interbank liabilities such as the total liability and total claim of a financial institution is available while the asset of a financial institution is typically subject to market shock. In other words, both the data matrix and the right hand side of the constraints in the underlying linear optimization model are uncertain. The presence of uncertainty in Eisenberg-Noe's model poses a tremendous challenge in estimating the contagious risk in the financial network and detecting the model's infeasibility.

Our first contribution in this work is to conduct a new extended sensitivity analysis to characterize the (in)feasibility of the LO model in Eisenberg and Noe (2001). To achieve such a goal, we propose to relax the LO model in Eisenberg and Noe (2001) by removing the non-negativity constraints in it. Then, we explore various properties of the relaxed model. Particularly, we derive a simple characterization for the feasibility of the relaxed model in terms of the summation of all the assets in the network (the total asset). Under the assumption that only a single shock is received by some bank in the network, we give a precise estimate on the amount of shock under which the receiving bank will be bankrupted, default or solvent. We also assess the impact of the single shock to all the non-receiving nodes in the system.

For the more generic scenario where all the banks are subject to market shocks, we first show that while a larger total asset may not improve the stability of a financial network, a larger asset inequality between a default node and a strictly solvent node in the network will reduce the stability of the network itself. To the best of our knowledge, our result is the first quantitative analysis showing that the asset inequality has a negative effect on the stability of the network. Then we study the network with a monopoly node where the monopoly owns an asset equals the total asset and dominates the entire network, representing an extreme scenario of the asset distribution. We show that the least stable asset distribution can be attained at some network with a monopoly node. We also estimate the probability of insolvency in the system under certain assumptions on the market shock and network structure, and show that the network with a monopoly node has the highest probability of insolvency and thus is the most vulnerable one. By using duality theory in linear optimization and stochastic optimization, we derive lower bounds for the probability of bankruptcy in the network. Particularly, we show that if the monopoly node in a financial network is liability free, then the probability of bankruptcy in the system is larger than 50%.

We also estimate the impact of bankruptcy in a financial network with a monopoly node. We first show that the bankruptcy of the monopoly node in a network will cause all other nodes in the network to be bankrupted, a catastrophical disaster to the entire system due to the domino effect of bankruptcy. In other words, the monopoly node is too big to fail (TBTF). We further explore the domino effect of bankruptcy in a financial network under a tridiagonal structure. We show that even when the monopoly node in such a financial network is solvent, the bankruptcy of every non-monopoly node in the network will cause other nodes following it to bankrupt consecutively.

The paper is organized as follows. In section 2, we first relax Eisenberg-Noe's model and explore various properties of the relaxation model. Then, we give a simple characterization on the (in)feasibility of the relaxation model. In section 3, we consider a scenario of the network where only a single node receives the shock. We first characterize different conditions under which the receiving node is solvent, default or bankrupted. We also develop a new algorithm to estimate the indirect impact of a single shock to the non-receiving nodes in the system. In section 4, we consider a scenario where all the nodes are subject to market shocks. We first study the impact of asset inequality on the stability of the system. Then, we estimate the probability that some bank in the network will be insolvent or be bankrupted under assumptions on the market shocks and network structure. In Section 5, we assess the impact of bankruptcy in a financial network with a monopoly node. Finally we conclude the paper in Section 6 by discussing some future research directions.

## 2. A Relaxation Model and Its Properties

As pointed out in the introduction, though many optimization models were proposed to measure the systemic risk of a financial network in Eisenberg and Noe (2001). To study the vulnerability of the network, it suffices to investigate the linear optimization model. Consider a financial network consisting of $n$ banks denoted by $n$ nodes where each bank borrows from one another and thus it owes liabilities to others. A clearing agent is in charge of the process of settling the liabilities among these nodes. The value of one node's payment to settle its obligations depends on the payment of other nodes to this node. Let $L \in \Re^{n\times n}$ be the interbank liability matrix where $l_{ij}$ is the liabilities of node $i$ toward node $j$. Since each nominal claim is nonnegative and no node has a nominal claim against itself, therefore we have $l_{ij} \geq 0$ and $l_{ii} = 0$, $\forall i, j = 1, ..., n$. Let $\alpha$ be the exogenous asset vector that subjects to market shocks. The total liabilities of node $i$ is equal to $\overline{p}_i = \sum_{j=1}^{n} l_{ij}$. In Eisenberg-Noe's model, the interbank payment made by node $i$ to node $j$ is $x_i l_{ij}$ which can be obtained by solving the following linear optimization problem:

$$\begin{aligned} \max \quad & \overline{p}^T x \\ s.t. \quad & (\overline{P} - L^T)x \leq \alpha, \\ & 0 \leq x \leq e, \end{aligned} \tag{2.1}$$

where $\overline{P}$ =diag($\overline{p}$) and $e$ is the all-ones vector. Note that numerous variants of model (2.1) have been discussed in Chen et al. (2016), Elsinger et al. (2013), Liu and Staum (2010). As proved in Eisenberg and Noe (2001) (see section 3.2 of Eisenberg and Noe (2001)), if all the coefficients in the objective function in these variants of model (2.1) are positive, then all these variants of the model have the same unique optimal solution. Therefore, for convenience we analyze only model (2.1), which has caught a lot of attention from various researchers in recent years and many results have reported in the literature. As pointed out in the introduction, many existing results focus on how the contagion risk spreads over the network when some bank in the network defaults, or when some bank in the network receives a single shock. For more details, we refer to recent works (Acemoglu et al. 2013, Capponi et al. 2016, Chen et al. 2016, Cont et al. 2010, Elsinger et al. 2005, 2006, 2013, Glasserman and Young 2015, Liu and Staum 2010) and the references therein.

In this paper, we try to estimate the impact of the market shock when all the banks are subject to market shocks from a certain distribution. We focus primarily on identifying conditions under which some banks in the network will be bankrupted or default. For such a purpose, we first suggest to remove the non-negativity constraint in (2.1), resulting in the following relaxation.

$$\begin{aligned} \max_x \quad & \bar{p}^T x \\ s.t. \quad & (\overline{P} - L^T)x \le \alpha; \\ & x \le e. \end{aligned} \tag{2.2}$$

Since we do not require all the variables to be nonnegative in model (2.2), it is possible that at the optimal solution of (2.2), there exists some index $i$ satisfying $x_i^* < 0$, which indicates that bank $i$ needs subsidies from other banks in the system to survive. In such a case, $|p_i x_i^*|$ represents the total subsidy bank $i$ received and $|l_{ij} x_i^*|$ represents the subsidy bank $i$ received from bank $j$. Recall that from a perspective of pure liability cleaning, the agent can also remove a bank $i$ from the network if the bank can not payback any of its liabilities. In such a case, the agent needs to solve only problem (2.1) with a reduced network. In comparison, an agent using model (2.2) aims at not only cleaning up the liabilities, but also helping the banks in crisis to survive with a minimal amount of subsidy.

Our next result establishes the equivalence between two problems (2.1) and (2.2) under the assumption that problem (2.1) is feasible.

PROPOSITION 2.1. *Let $x^{(1)}$ be the optimal solution of problem (2.1) and $x^{(2)}$ be the optimal solution of the problem (2.2). Then we have*

$$x^{(2)} = x^{(1)}.$$

*Proof.* The proof of the proposition is a minor modification of the proof of Theorem 2.1 in Pokutta et al. (2011). For self-completeness, we give the detailed proof here. Note that to prove the above proposition, it suffices to show that $x^{(2)} \ge 0$. Let $X^1, X^2$ be the feasible set to problems (2.1) and (2.2), respectively. Clearly both $X^1$ and $X^2$ are bounded and convex. Moreover, we have

$X^1 \subset X^2$. Since problem (2.1) is feasible and thus, $X^1$ is nonempty, it follows that $X^2$ is also nonempty. For the bounded nonempty set $X^2$, let $\bar{x}$ be the vector whose element $\bar{x}_i, i=1,\cdots,n$ is defined by

$$\bar{x}_i := \sup_{x\in X^2} x_i. \tag{2.3}$$

It follows that for every $i=1,\cdots,n$, we have

$$\bar{x}_i := \sup_{x\in X^2} x_i \geq \sup_{x\in X^1} x_i \geq 0,$$

where the inequality follows from the fact $X^1 \subset X^2$.

We next show that $\bar{x}$ is the unique optimal solution of problem (2.2). For every $i \in \{i=1,2,...,n\}$, there is $x^i \in X^2$ such that $x^i_i = \bar{x}_i$, thus

$$\bar{p}_i\bar{x}_i - \sum_j l_{ji}x^i_j = \bar{p}_i x^i_i - \sum_j l_{ji}x^i_j$$

holds. Because $l_{ji} \geq 0$, we have

$$\bar{p}_i\bar{x}_i - \sum_j l_{ji}\bar{x}_j \leq \bar{p}_i\bar{x}_i - \sum_j l_{ji}x^i_j,$$

and we can conclude by the fact $x^i \in X^2$ that

$$\bar{p}_i\bar{x}_i - \sum_j l_{ji}\bar{x}_j \leq \alpha_i$$

i.e, $\bar{x}$ is a feasible solution to problem (2.2). Clearly, $\bar{x}$ maximizes $\sum_i \bar{p}_i x_i$ and therefore, $\bar{x}$ is the unique optimal solution to (2.2). This completes the proof of the proposition. □

As discussed in Eisenberg and Noe (2001), the optimal solution of problem (2.1) is a clearing vector for the financial system that satisfies the so-called limited liability and absolute priority. By following a similar process as in Eisenberg and Noe (2001), we can obtain the following result.

Corollary 2.1. *At the optimal solution of problem (2.2), we have either*

$$[(\overline{P} - L^T)x^*]_i = \alpha_i \text{ or } x^*_i = 1, \ \forall i=1,\cdots,n.$$

From Propositions 2.1 and Corollary 2.1 we can see that there is no differences regarding the properties of the optimal solutions to problems (2.1) and (2.2), and the optimal solution of problem (2.2) can also be used as a clearing vector for the financial system. However, as we shall see in our later analysis, a key difference between problems (2.1) and (2.2) lies in the fact that a simple characterization of the (in)feasibility of problem (2.2) can be derived, which further facilitates the feasibility analysis of model (2.1). Before stating our main result in this section, we need the following definition

DEFINITION 2.1. Two nodes $i$ and $j$ in the financial network are said to be connected if there exists a path $i = i_0 \to i_1 \to \cdots \to i_{K-1} \to i_K = j$ such that

$$l_{i_{k-1} i_k} > 0, \quad \forall k = 1, \cdots, K.$$

A financial network is said to be fully connected if every pair of nodes in the network is connected. For convenience, we also say a financial network is well-balanced if the following relation holds.

$$\bar{p}_i = (L^T e)_i, \quad \forall i = 1, \cdots, n.$$

In the paper, we make the following assumption:

ASSUMPTION 2.1. *The financial network is fully connected and well-balanced.*

We remark that Assumption 2.1 is rather mild because if a well-balanced financial network is not fully connected, then we can divide it into two independent subnetworks such that there exists no connections between these two subnetworks. Correspondingly, we can solve two smaller linear optimization problems to obtain the clearing vectors for these two independent systems.

We next present a technical result that will be used in the analysis later on.

LEMMA 2.1. *Suppose that the financial network is fully connected. Let $\lambda^*$ be a solution of the following system of linear inequalities*

$$(\overline{P} - L)\lambda \le 0, \;\; \lambda \ge 0. \tag{2.4}$$

*Then, it must hold $\lambda^* = ce$ for some $c \ge 0$.*

*Proof.* Without loss of generality, we assume $\lambda \ne 0$. Let $\lambda_{i*} = \max_{i=1,\cdots,n} \lambda_i$, and define the index set $I_{i*} = \{j \ne i* : l_{i*j} > 0\}$. From the definition of $\bar{p}_{i*}$, we have $\bar{p}_{i*} - \sum_{j \in I_{i*}} l_{i*j} = 0$. It follows from (2.4) that

$$0 \ge \bar{p}_{i*}\lambda_{i*} - \sum_{j \in I_{i*}} l_{i*j}\lambda_j \ge \lambda_{i*}(\bar{p}_{i*} - \sum_{j \in I_{i*}} l_{i*j}) = 0. \tag{2.5}$$

From (2.4) and (2.5) we can conclude

$$\lambda_j = \lambda_{i*} = \max_{i=1,\cdots,n} \lambda_i, \quad \forall j \in I_{i*}.$$

Similarly, for every $i \in I_{i*}$ and the index set $I_i = \{j \ne i : l_{ij} > 0\}$, we have

$$\lambda_j = \lambda_{i*}, \quad \forall j \in I_i. \tag{2.6}$$

Now let us choose arbitrary any index $j*$. Recall that the network is fully connected. Let us first consider a case where there exists a path $i* = i_1 \to i_2 \cdots \to i_{K-1} \to i_k = j*$. By following a similar vein as in the proof of (2.6), we can conclude $\lambda_{j*} = \lambda_{i*}$. This completes the proof of the lemma. □

Now we are ready to state the main result in this section.

THEOREM 2.1. *Suppose that Assumption 2.1 hold. Then problem (2.2) is infeasible if and only if*

$$\sum_i \alpha_i < 0. \tag{2.7}$$

*Proof.* First, we note that the dual problem of (2.2) reads as

$$\begin{aligned} \min_{\lambda} \quad & \alpha^T \lambda + e^T \bar{p} - e^T (\overline{P} - L)\lambda \\ s.t. \quad & (\overline{P} - L)\lambda \leq \bar{p}; \\ & \lambda \geq 0. \end{aligned} \tag{2.8}$$

Using the Farkas Lemma, we see that problem (2.2) is infeasible if and only if the following linear system is feasible.

$$\left\{ \begin{array}{c} \alpha^T \lambda - e^T (\overline{P} - L)\lambda < 0; \\ (\overline{P} - L)\lambda \leq 0, \lambda \geq 0. \end{array} \right\} \tag{2.9}$$

It remains to show that the system (2.9) has a nontrivial solution if and only if (2.7) holds.

If $e^T \alpha < 0$, from the definition of liability matrix $L$ we have $(\overline{P} - L)e = 0$. This implies that for any $c > 0$, $ce$ is a feasible solution of the system (2.9), and therefore, problem (2.2) is infeasible.

On the other hand, suppose that the system (2.9) has a nontrivial feasible solution. From Lemma 2.1 we can conclude that $\lambda = ce$ for some $c \geq 0$. It follows from (2.9) that $c\alpha^T e < 0$, and thus it must hold $\alpha^T e < 0$. This completes the proof of the theorem. □

Theorem 2.1 indicates that we can use the total asset as an indicator for the infeasibility of the relaxed problem (2.2) and such an indicator is independent of the liability matrix. We next explore the financial meaning of Theorem 2.1. For this, let us consider a scenario where the clearing agent has a superpower to redistribute the assets of all the banks in the system. Under the assumption that the total asset is fixed, the agent would like to use its power to maximize its revenue, leading to the following two-stage linear optimization problem

$$\begin{aligned} \max_{\alpha} \max_{x} \quad & \bar{p}^T x \\ s.t. \quad & (\bar{P} - L^T)x \leq \alpha; \\ & x \leq e; \\ & \sum_i \alpha_i = \bar{\alpha}. \end{aligned} \tag{2.10}$$

We have

Proposition 2.2. *If the total asset is non-negative ($\bar{\alpha} \geq 0$), then the optimal solution to problem (2.10) can be attained at some $\alpha^*$ such that $x_i^*(\alpha^*) = 1, \ \forall i = 1, \cdots, n$.*

*Proof.* We first consider the special case when all the nodes in the system are well-balanced. In this case, we have $(\bar{P} - L^T)e = 0$. This implies that for every asset vector $\alpha^*$ satisfying $\alpha_i^* \geq 0, \ \forall i = 1, \cdots, n$, we have $x^*(\alpha^*) = e$ and thus, all the nodes in the system are solvent.

Next we consider the generic case $\sum_i \bar{p}_i = \sum_i (L^T e)_i$. In this case, one can show that if we choose the asset vector $\alpha^*$ by $\alpha_i^* = \bar{\alpha}/n + \bar{p}_i - (L^T e)_i, \ \forall i = 1, \cdots, n$, then we still have $x^*(\alpha^*) = e$. This shows that all the nodes in the system are solvent.

From Theorem 2.1 and Proposition 2.2 we can conclude that the value of the total asset $\bar{\alpha}$ indicates whether we can redistribute the assets to make all the nodes in the network solvent or there exists no ways to make all the nodes solvent. Since problem (2.2) is a relaxation of problem (2.1), the condition $\bar{\alpha} < 0$ can also be viewed as a sufficient condition for the infeasibility of problem (2.1). It is interesting to note that Theorem 2.1 holds true for the generic class of financial networks that are fully connected, which shows that the vulnerability of the financial network may not depend on the liability matrix $L$.

Our next theorem explore various properties at the optimal solution of problem (2.2).

Theorem 2.2. *Suppose that the total asset of the financial network is nonnegative ($\bar{\alpha} = \alpha^T e \geq 0$). Let $x^*$ be the optimal solution of problem (2.2). Then the following conclusions hold.*

*(i) There exists at least one index $i$ such that $x_i^* = 1$;*

*(ii) For every $i = 1, \cdots, n$, if $\alpha_i < \bar{p}_i - (L^T e)_i$, then $x_i^* < 1$;*

*(iii) For every $i = 1, \cdots, n$, if $x_i^* = 1$, then $\alpha_i \geq \bar{p}_i - (L^T e)_i$;*

*Proof.* We start with the proof of Conclusion (i). Let us first consider the case when $\bar{\alpha} > 0$. Suppose to the contrary that Conclusion (i) does not hold, i.e., at the optimal solution $x^*$ of problem (2.2), we have

$$x_i^* < 1, \quad \forall i = 1, \cdots, n.$$

From Proposition 2.1, the following condition holds

$$(\overline{P} - L^T)x^* = \alpha.$$

Because $e^T(\overline{P} - L^T) = 0$, it follows that $e^T \alpha = \bar{\alpha} = 0$, which contradicts to the assumption $\bar{\alpha} > 0$. To prove the conclusion when $\bar{\alpha} = 0$, let us define a new vector $\alpha^\epsilon$ where

$$\alpha_1^\epsilon = \alpha_1 + \epsilon, \quad \alpha_i^\epsilon = \alpha_i, \forall i = 2, \cdots, n.$$

It is easy to see that

$$e^T \alpha^\epsilon = \bar{\alpha} + \epsilon > 0, \quad \forall \ \epsilon > 0.$$

Now let us consider a variant of problem (2.2) where $\alpha$ is replaced by $\alpha^\epsilon$ and let us denote its optimal solution by $x^*(\epsilon)$. Since $e^T\alpha^\epsilon > 0$, there exists some index $i$ such that $x_i^*(\epsilon) = 1$. Now let us choose a sequence of $\epsilon^k \to 0$. By restricting us to a subsequence if necessary, we can see that there exists an index $i$ such that $x_i^*(\epsilon^k) = 1$. Recall that problem (2.2) is a linear program, by using the continuity of the solution sets for linear programs (Mangasarian and Shiau 1987), we have

$$x_i^* = x_i^*(0) = \lim_{k\to\infty} x_i^*(\epsilon^k) = 1.$$

This completes the proof of conclusion (i).

To prove the second conclusion, we note that at $x^*$, the following inequality

$$\bar{p}_i x_i^* - \sum_{j\neq i} l_{ji} x_j^* \leq \alpha_i, \quad \forall\ i = 1, \cdots, n,$$

holds. Therefore, for every $i = 1, \cdots, n$, we have

$$\bar{p}_i x_i^* \leq \alpha_i + \sum_{j\neq i} l_{ji} x_j^* \leq \alpha_i + \sum_{j\neq i} l_{ji} = \alpha_i + (L^T e)_i < \bar{p}_i, \tag{2.11}$$

where the second inequality follows from the constraint $x^* \leq e$, and the last inequality from the assumption $\alpha_i < \bar{p}_i - (L^T e)_i$. From (2.11) we immediately obtain

$$x_i^* < 1.$$

The third conclusion follows directly from Conclusion (ii). This completes the proof of the theorem. □

Conclusion (i) in Theorem 2.2 shows that when the total asset is non-negative, at least one bank remains solvent in the system. Based on the second conclusion in Theorem 2.2, we can estimate the upper bound for the asset value of bank $i$ under which bank $i$ will be insolvent in the system. We note this upper bound can be obtained from the current market information.

We remark that though the results of Theorem 2.2 are established for problem (2.2), from Proposition 2.1 one can easily see that these results also hold true for problem (2.1) when it is feasible. We note that similar results like Conclusions (ii) and (iii) in Theorem 2.2 have been obtained in Chen et al. (2016) where the authors used them to design an effective algorithm, and estimate the impact of a single shock to the system based on the classical sensitivity analysis in linear optimization and probability theory. In this paper, we shall use these results to estimate the impact of market shocks on the financial network.

We next provide a numerical example to verify the theoretical results in this section.

**Table 1 An Example for the Relaxed Model of Financial Network.**

| Liability Matrix | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Node | 1 | 2 | 3 | 4 | $\bar{p}$ | $\alpha^1$ | $\alpha^{1'}$ | $\alpha^2$ |
| 1 | 0 | 4857 | 9971 | 11306 | 26134 | -6700 | -6700.1 | -9400 |
| 2 | 4857 | 0 | 10625 | 12047 | 27529 | 1790 | 1790 | 32790 |
| 3 | 9971 | 10625 | 0 | 24734 | 45330 | 740 | 740 | -8100 |
| 4 | 11306 | 12047 | 24734 | 0 | 48087 | 4170 | 4170 | 12358.2 |
| Claims | 26134 | 27529 | 45330 | 48087 | 147080 | 0 | -0.1 | 27648.2 |

| Optimal Solution | | | | |
|---|---|---|---|---|
| $x_1^*$ | $x_2^*$ | $x_3^*$ | $x_4^*$ | |
| 0.7269 | 1 | 0.9562 | 1 | $\alpha^1$ |
| infeasible | - | - | - | $\alpha_1^{1'} = -6700.1$ |
| .5329 | 1 | 0.7186 | 1 | $\alpha_{1-}^2 = -9400$ |
| -1.2097 | 1 | -0.1232 | 0.1597 | $\alpha_{1+}^2 = -37048.2$ |

Example 2.1. We consider a complete financial network with four banks in which the total asset equals zero ($\bar{\alpha} = e^T\alpha = 0$). The liability matrix is extracted from the liability matrix (see Table 6 in Chen et al. (2016)) by considering the first four banks in the network. The asset vector and the optimal solution are also listed below.

We point out that all the linear optimization problems in the numerical experiments of this work are solved by using CVX Grant and Boyd (2013) under MATLAB R2015b. In Example 2.1, we first consider an asset vector $\alpha^1$ satisfying $\bar{\alpha}^1 = 0$. Note that the network is well-balanced. From Table 1 one can see that both nodes 2 and 4 can fulfill their liabilities while nodes 1 and 3 default. This is consistent with the conclusions in Theorem 2.2. Based on the optimal solution, we see that even bank 1 can payback more than 70% of its liability and thus it is not easy to determine whether there exists a high risk of bankruptcy in the system. However, because $\bar{\alpha}^1 = 0$, from Theorem 2.1 we can conclude that if some bank $i$ in the network receives a negative shock, then both problems (2.1) and (2.2) will become infeasible, indicating a high potential risk of bankruptcy in the system. To verify such a result, we reduce the value of $\alpha_1$ slightly to $\alpha_1^{1'} = -7600.1$ and thus $\bar{\alpha}^{1'} < 0$. As one can see from the table, problem (2.2) becomes infeasible.

We also consider problem (2.2) with another asset vector $\alpha^2$ satisfying $\bar{\alpha}^2 > 0$. By reducing the value of $\alpha_{1-}^2$ to $\alpha_{1+}^2 = -37048.2$, we obtain a new total asset value 0, and thus, problem (2.2) remains feasible. We observe that at the optimal solutions, we have $x_1^2 < 0$ and $x_3^2 < 0$, which indicates both banks 2 and 3 need to be subsidized to survive. By checking the optimal solutions again, we can also find that because of the change in $\alpha_1^2$, the solvent node (4) has changed to a default one ( $x_4^*(\alpha_{1-}^2)=1$ and $x_4^*(\alpha_{1+}^2)=0.1597<1$). This demonstrates that in such a case, the small shock assumption widely used in the literature of systemic risk analysis does not hold.

## 3. The Vulnerability of A Financial Network under A Single Shock

In this section, we consider a scenario where only a single node receives the market shock and study the impact of the shock on the whole system. The section consists of two subsections. In the first subsection, we estimate the direct impact of the shock on the receiving node, and in the second subsection we study the indirect impact of the shock on other nodes in the system.

### 3.1. The impact of a single shock on the receiving node

In this subsection, we consider a scenario where only a single node receives the market shock and present a deterministic characterization on conditions under which the receiving node ($i$) will be solvent, default or bankrupted. For such a purpose, we consider the following modified variant of problem (2.2)

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} \bar{p}_i x_i \\ s.t. \quad & (\bar{P}-L^T)x \leq \alpha + s; \\ & x \leq e, \end{aligned} \tag{3.1}$$

where $s$ has only a single nonzero element $s_i \neq 0$ at some index $i$. We can interpret $s$ as the influence of the market shock on bank $i$. Our purpose is to characterize the behavior of the optimal solution $x^*(s)$ of problem (3.1) in terms of $s$. Particularly, we are mainly interested in conditions under which the $i$-th element $x_i^*(s)$ at the optimal solution will satisfy one of the following conditions:

$$x_i^*(s) \leq 0; \tag{3.2}$$

$$x_i^*(s) \in (0,1); \tag{3.3}$$

$$x_i^*(s) = 1. \tag{3.4}$$

Note that the above conditions are associated with the status of the bank $i$ depending on whether it is bankrupted, default, or solvent. We also call node $i$ insolvent when $x_i^*(s)<1$. Next, we first present a technical result.

LEMMA 3.1. *Suppose that $x^*(s)$ be the optimal solution of problem (3.1) where $s$ has only a single nonzero element $s_i \neq 0$ for some $i$. Then the following conclusions hold.*

*(i) For every index $j \in \{1,\cdots,n\}$, $x_j^*(s)$ is nondecreasing in terms of $s_i$;*

*(ii) For index $i$, $x_i^*(s)$ is locally strictly increasing in terms of $s_i$ if $x_i^*(s)<1$;*

*Proof.* We start with the proof of Conclusion (i). Let $x^*(s^1)$ and $x^*(s^2)$ denote the optimal solutions of problem (3.1) corresponding to $s^1$ and $s^2$ respectively, where $s_i^1 < s_i^2$. We want to show that $x_j^*(s^1) \leq x_j^*(s^2)$ for every index $j \in \{1, \cdots, n\}$. Assume to the contrary that there is an index $k$ such that $x_k^*(s^1) > x_k^*(s^2)$. Since we assume that $s^1 < s^2$, it follows that

$$X_s^1 \subseteq X_s^2, \tag{3.5}$$

where $X_s^1$ and $X_s^2$ are the feasible sets corresponding to $s^1$ and $s^2$ respectively. According to the proof of Proposition 2.1, for every index $i$, we have

$$x_i^*(s) = \sup_{x \in X_s} x_i. \tag{3.6}$$

where $X_s$ is the feasible set of problem (3.1). From (3.5) and (3.6) we can conclude that $x_k^*(s^1) \leq x_k^*(s^2)$ which contradicts the assumption and finishes the proof of the first conclusion.

To prove the second conclusion it suffices to show that for $s_i^1 < s_i^2$, $x_i^*(s^1) \neq x_i^*(s^2)$. Suppose to the contrary that $x_i^*(s^1) = x_i^*(s^2)$. Now let us define the index sets

$$\mathcal{I}_1 = \{i : \bar{p}_i x_i^* - \sum_{j \neq i} l_{ji} x_j^* = (\alpha + s)_i\}, \quad \mathcal{I}_2 = \{i : x_i^* = 1\}.$$

From Proposition 2.1, we can conclude that $x^*$ is the unique solution of the following linear equation system

$$\bar{p}_i x_i^* - \sum_{j \neq i} l_{ji} x_j^* = (\alpha + s)_i, \ \forall i \in \mathcal{I}_1; \tag{3.7}$$

$$x_i^* = 1, \quad \forall i \in \mathcal{I}_2. \tag{3.8}$$

Let us denote the coefficient matrix in the above system by $A$. Clearly $A$ is a so-called $M$-matrix. On the other hand, since $\mathcal{I}_2$ is nonempty by Theorem 2.2, under the assumption that the financial network is fully connected, one can show that $A$ is nonsingular. Since we assume that $x_i^*(s) < 1$, we have

$$x^*(s_i) = A^{-1}\alpha + s_i A_{:i}^{-1},$$

where $A_{:i}^{-1}$ is the $i$-th column of $A^{-1}$. Note that the inverse matrix of an $M$-matrix is nonnegative (see Berman and Plemmons (1979)). Using the fact that $A_{ii}$ is the only positive element in the $i$-th row of $A$ and $A^{-1}$ is nonnegative, we can conclude $A_{ii}^{-1} > 0$. This implies that for $s_i^1 < s_i^2$, $x_i^*(s^1) < x_i^*(s^2)$ which contradicts to the assumption. This completes the proof of the lemma. □

Now we are ready to state the main result in this section.

THEOREM 3.1. *Let $x^*(s)$ be the optimal solution of problem (3.1). Then, $x_i^*(s) < 0$ if and only if*

$$s_i < \max(-\bar{\alpha}, \Delta_i - \alpha_i), \tag{3.9}$$

*where*

$$\begin{aligned} \Delta_i = -\max \ & \sum_{j \neq i} l_{ji} x_j \\ s.t.\ & \bar{p}_j x_j - \sum_{k \neq j, k \neq i} l_{kj} x_k \leq \alpha_j,\ \forall j \neq i; \\ & x_j \leq 1,\ \forall j \neq i. \end{aligned} \tag{3.10}$$

*Proof.* For simplicity, we consider only the special case $s_1 \neq 0$. To prove the sufficiency of the theorem, we consider the following two cases: Case (i): $s_1 < -\bar{\alpha}$. Case (ii): $-\bar{\alpha} \leq s_1 < \Delta_1 - \alpha_1$. In the first case, we have

$$e^T \alpha + s_1 < 0.$$

It follows from Theorem 2.1 that problem (3.1) is infeasible which implies that problem (2.1) is also infeasible.

Now we consider case (ii) where $-\bar{\alpha} \leq s_1 < \Delta_1 - \alpha_1$. In such a case, we have $\bar{\alpha} + s_1 \geq 0$ and thus the relaxed problem (3.1) is feasible. It remains to show that $x_1^* < 0$ at the optimal solution $x^*$ of problem (3.1). Let us consider the following specific variant of problem (3.1)

$$\begin{aligned} \max \ & \sum_{i=1}^{n} \bar{p}_i x_i \\ s.t.\ & \bar{p}_1 x_1 - \sum_{j \neq 1} l_{j1} x_j \leq \Delta_1; \\ & \bar{p}_i x_i - \sum_{j \neq i} l_{ji} x_j \leq \alpha_i,\ \forall i = 2, \cdots, n; \\ & x \leq e. \end{aligned} \tag{3.11}$$

One can verify that the above problem and problem (3.10) have the same optimal solution, which further implies $x_1^*(s_1) = 0$ when $s_1 = \Delta_1 - \alpha_1$. According to Conclusion (ii) in Lemma 3.1 we can conclude that $x_1^*(s_1)$ is locally strictly increasing in terms of $s_1$. Therefore, we have $x_1^*(s_1) < 0$ whenever $s_1 < \Delta_1 - \alpha_1$. This proves the sufficiency of the theorem.

Now we consider the necessity of the theorem. It suffices to consider the case where problem (3.1) is feasible and $x_1^*(s_1) < 0$. Suppose to the contrary that the relation (3.9) does not hold, i.e.,

$$s_1 \geq \max(-\bar{\alpha}, \Delta_1 - \alpha_1). \tag{3.12}$$

The above relation indicates

$$s_1 + \bar{\alpha} \geq 0, \quad s_1 + \alpha_1 \geq \Delta_1. \tag{3.13}$$

Recall $x_1^*(\Delta_1 - \alpha_1) = 0$, from Lemma 3.1 we immediately obtain

$$x_1^*(s_1) \geq 0,$$

which contradicts to the assumption. This completes the proof of the theorem. □

We remark that $|\Delta_i|$ represents the maximum amount of repayments that bank $i$ received from other banks in the system under the condition that bank $i$ does not make any payment to other banks in the system, i.e., $x_i^* = 0$. Theorem 3.1 provides an estimate on the amount of negative shock that a financial institution can survive bankruptcy under the condition that only the corresponding institution is affected by the shock. We also mention that the estimated amount of negative shock in Theorem 3.1 depends only on the current market information, not on the future market fluctuation. Such a result allows the financial institution to estimate the worst-case scenario it can survive under the current market conditions on the assets and interbank liabilities, which may help the financial institution in its decision making to hedge future risk.

We next present a numerical example to verify the theoretical conclusions of Theorem 3.1.

EXAMPLE 3.1. Consider a network of four financial institutions where the liability matrix is the same as in Example 2.1. The asset vector $\alpha = (-9400, 32790, -8100, 12358.2)^T$ with $\bar{\alpha} = e^T\alpha = 27648.2 > 0$.

**Table 2 Bankruptcy Example in Financial Network**

| Optimal Solution | | | | |
|---|---|---|---|---|
| $x_1^*$ | $x_2^*$ | $x_3^*$ | $x_4^*$ | |
| 0.0001 | 1 | 0.46 | 0.75 | $\alpha_1 = -17893$ |
| 0 | 1 | 0.46 | 0.75 | $\alpha_1 = \Delta_1 = -17894.736$ |
| -0.0001 | 1 | 0.46 | 0.75 | $\alpha_1 = -17896$ |

For the above example, we estimate the maximum amount of negative shock ($s$) via (3.9) that a node in the network can sustain to survive, where $\Delta$ is obtained by solving problem (3.10). The results are shown in the following table.

**Table 3 $\Delta$ and $s$ estimated for Example 3.1**

| Node | $\Delta$ | $s$ |
|---|---|---|
| 1 | -17894.736 | -8494.736 |
| 2 | 5141.80 | -27648.20 |
| 3 | -23981.652 | -15881.652 |
| 4 | -10635.42 | -22993.62 |

For example, based on Table 3, it holds $s_1 = -8494.736$. From Theorem 3.1, we can conclude that if $s_1 < -8494.736$, then at the optimal solution we have $x_1^* < 0$. To verify that, we consider three different shocks with $s_1 = -8493 > -8494.736$, $s_1 = -8494.736$, and $s_1 = -8496 < -8494.736$, respectively. Correspondingly we obtain three different values for asset ($\alpha_1$) as listed in Table 2. As one can see from Table 2, in all these cases, we have $\bar{\alpha} > 0$ and thus, problem (2.2) is feasible. However, we note that when $s_1 = -8496$ ( or equivalently $\alpha_1 = -17896$), it holds $x_1^* = -0.0001 < 0$, which indicates problem (2.1) is infeasible.

We next compare Theorem 3.1 with the results in Liu and Staum (2010) where they used standard sensitivity analysis to estimate the impact on the payments with respect to some changes in the asset vector (called the partial derivative of the repayments with respect to the assets). As shown in Eisenberg and Noe (2001), problem (2.1) can be solved via solving $n$ decomposed problems where the objective is to maximize the payment for every node $i$ subject to the same constraint set as in the original problem (2.1) [1]. Therefore, the partial derivatives of the repayments with respect to the assets are precisely the shadow prices for the decomposed problems. The matrix of shadow prices computed in Liu and Staum (2010) is given as follows

$$\frac{\partial p^*}{\partial \alpha} \approx \begin{bmatrix} 1.09 & 0 & 0.41 & 0 \\ 0 & 0 & 0 & 0 \\ 0.24 & 0 & 1.09 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $p_i^* = x_i^* * \bar{p}_i$. For example, based on the above estimate, we have that $\frac{\partial p_1^*}{\partial \alpha_1} \approx 1.09$ indicating that a decrease of \$1 in asset value of the first node cause the payment made by that node to drop approximately \$1.09. Unfortunately, such a result remains valid only for small shocks under which both the set of default nodes and the set of solvent nodes remain invariant, and the result will not hold any more for reasonably large shocks that may cause a solvent node to become a default one. Take for example, for the solvent node 4, we have $\frac{\partial p_4^*}{\partial \alpha_4} \approx 0$ whenever the node remains solvent. This illustrates that the results in Liu and Staum (2010) cannot be used to predict when a node will become bankrupted as a large negative shock to the asset vector may change the sets of default and solvent nodes. In contrast, Theorem 3.1 shows that if node 4 receives a large negative shock $s_4 = -22993.62$, then node 4 will be bankrupted.

In what follows we study the solvency of a financial institution in the system, i.e., to characterize when $x_i^*(s) = 1$ for a given index $i$.

THEOREM 3.2. *Suppose that $x^*(s)$ be the optimal solution of problem (3.1). Then we have $x_i^*(s) = 1$ if and only if*

$$s_i \geq \bar{p}_i + \Gamma_i - \alpha_i. \tag{3.14}$$

*where*

$$\begin{aligned}\Gamma_i = -\max \ & \sum_{j\neq i} l_{ji}x_j \\ s.t. \ \ & \bar{p}_j x_j - \sum_{k\neq j, k\neq i} l_{kj}x_k \leq \alpha_j + l_{ij}, \ \forall j\neq i; \\ & x_j \leq 1, \ \forall j \neq i.\end{aligned} \tag{3.15}$$

*Proof.* Let us consider only the special case $s_1 \neq 0$. To prove the sufficiency of the theorem, we assume that condition (3.14) holds. Since $s_1 = \bar{p}_1 + \Gamma_1 - \alpha_1$, we can rewrite problem (3.1) as follows

$$\begin{aligned}\max \ & \sum_{i=1}^{n} \bar{p}_i x_i \\ s.t. \ \ & \bar{p}_1 x_1 - \sum_{j\neq 1} l_{j1}x_j \leq \bar{p}_1 + \Gamma_1; \\ & \bar{p}_i x_i - \sum_{j\neq i} l_{ji}x_j \leq \alpha_i, \ \forall i = 2, \cdots, n; \\ & x \leq e.\end{aligned} \tag{3.16}$$

One can easily verify that problem (3.16) and (3.15) have the same optimal solution, which implies that we have $x_1^*(s_1) = 1$ if $s_1 = \bar{p}_1 + \Gamma_1 - \alpha_1$. From Conclusion (i) of Lemma 3.1 we know that $x^*(s_1)$ is nondecreasing in terms of $s_1$. Therefore, we have $x_1^*(s_1) = 1$ for $s_1 \geq \bar{p}_1 + \Gamma_1 - \alpha_1$. This proves the sufficiency of theorem.

Now we consider the necessity of the theorem. Suppose that $x_1^*(s_1) = 1$. Suppose to the contrary that inequality (3.14) does not hold, i.e.,

$$s_1 < \bar{p}_1 + \Gamma_1 - \alpha_1.$$

From the constraints of problem (3.1) we obtain

$$\bar{p}_i - \sum_{j\neq i} l_{ji}x_j^* = \bar{p}_i x_i^* - \sum_{j\neq i} l_{ji}x_j^* \leq \alpha_i + s_i, \tag{3.17}$$

where the equality follows from the assumption that $x_i^*(s_i) = 1$. Therefore, we have

$$s_i \geq \bar{p}_i + \Gamma_i - \alpha_i,$$

which contradicts to the assumption. This finishes the proof of the theorem. □

We remark that $|\Gamma_i|$ denotes the maximum amount of repayments that a solvent bank $i$ received from other banks in the system. Theorem 3.2 provides an estimate on the minimum amount of positive shock or market gain for a default bank to become solvent. We mention that the estimated positive shock in Theorem 3.2 depends on the current asset value and interbank liabilities. This allows the banks to estimate the asset increment (such as the exogenous investment or profit gain) they need to become solvent. Theorem 3.2 also provides an estimate on the maximum amount of negative shock that a solvent node can sustain to remain solvent. The results in Theorem 3.2 may help banks in their decision making regarding their investment in the market. We also point out that the results in Theorem 3.2 are very different from the results in Liu and Staum (2010) based on standard sensitivity analysis in linear optimization.

We next use the same financial network as in Example 3.1 to verify the conclusions in Theorem 3.2.

EXAMPLE 3.2. $L$ and $\alpha$ are the same as in Example 3.1.

**Table 4 Solvency Example in Financial Network**

| Optimal Solution | | | | |
|---|---|---|---|---|
| $x_1^*$ | $x_2^*$ | $x_3^*$ | $x_4^*$ | |
| 0.99 | 1 | 0.82 | 1 | $\alpha_1 = 1780$ |
| 1 | 1 | 0.82 | 1 | $\alpha_1 = \Gamma_1 + \bar{p}_1 = 1781.714$ |
| 1 | 1 | 0.82 | 1 | $\alpha_1 = 1782$ |
| 0.53 | 1 | 0.71 | 1 | $\alpha_2 = 5258.652$ |
| 0.53 | 0.99 | 0.71 | 1 | $\alpha_2 = 5257$ |

**Table 5 $\Gamma$ and $s$ estimated for Example 3.2**

| Node | $\Gamma$ | $s$ |
|---|---|---|
| 1 | -24352.286 | 11181.714 |
| 2 | -22270.348 | -27531.348 |
| 3 | -41743.584 | 11686.416 |
| 4 | -35845.652 | -116.852 |

In the above example, $\Gamma$ is computed by solving problem (3.15). For this example, if a default node in the system, i.e., node 1, can manage to increase its asset, then its repayment ability will be improved. In the case that the increment of asset value reaches the target value ($s_1 = 11181.714$), then node 1 becomes solvent. To verify such a result, we consider three different shocks with $s_1 = 11180 < 11181.714$, $s_1 = 11181.714$, and $s_1 = 11182 > 11181.714$, receptively. The corresponding values of the asset $\alpha_1$ are listed in Table 4. As one can see from Table 4 that when $\alpha_1 \geq 1781.714$, it holds $x_1^* = 1$ at the optimal solution.

We also estimate the maximal amount of negative shock that a solvent node in the system can sustain to keep its solvency. Let us consider the solvent node 2 in the above example. Based on

Table 5, we have $s_2 = -27531.348$. From Theorem 3.2 we can conclude that if $s_2 \geq -27531.348$, we have $x_2^* = 1$. To verify such a result, we consider two shocks $s_2 = -27531.348$ and $s_2 = -27532$ respectively, and the corresponding asset values $\alpha_2 = 32790 - 27531.348 = 5258.652$ and $\alpha_2 = 32790 - 27533 = 5257$. As one can see from Table 4, it holds $x_2^* = 1$ when $s_2 = -27531.348$, and $x_2^* < 1$ when $s_2 = -27533$.

It is also interesting to note that standard sensitivity analysis for linear optimization problem can also be used in some special case in Theorem 3.2. Take for example, using the standard sensitivity analysis as in Liu and Staum (2010), we have $\frac{\partial p_1^*}{\partial \alpha_1} \approx 1.09$ (see discussion in Example 3.1). This shows that when $\alpha_1$ is increased by 11182.7158, the first node can repay all its liabilities. We note that in such a case, it holds

$$[(\overline{P} - L^T)x^*]_1 = \alpha_1, \qquad x_1^* = 1.$$

The above example represents a borderline case where node 1 changes from a default node to a solvent one. In other words, the sets of the default and solvent nodes did not change before $\alpha_1$ reaches the value 11182.7158. Therefore, the estimate based on the standard analysis is similar to what we obtained from Theorem 3.2. For solvent nodes 2 and 4, from the standard sensitivity analysis we have $\frac{\partial p_2^*}{\partial \alpha_2} \approx 0$ and $\frac{\partial p_4^*}{\partial \alpha_4} \approx 0$, which implies that the asset value of those nodes does not have any impact on their repayment ability. In contrast, Theorem 3.2 shows that if $\alpha_2 < 5258.651$, then node 2 will default.

Theorems 3.1 and 3.2 provide an upper bound and a lower bound of the magnitude of shock $s_i$ such that bank $(i)$ is bankrupted or solvent. Combining the results in these two theorems, we immediately obtain the following result.

COROLLARY 3.1. *If*

$$s_i \in (\max(-\bar{\alpha}, \Delta_i - \alpha_i), \bar{p}_i + \Gamma_i - \alpha_i), \tag{3.18}$$

*then the relation* $0 < x_i^*(s_i) < 1$ *holds at the optimal solution* $x^*(s_i)$ *of problem (3.1).*

We call interval (3.18) as the default window. The default window can be used as an indicator for the resistance of a default bank $i$ to market shock. The larger the window is, more resistant to market shock the bank is. It worths estimate the length of the default window. Since $\Delta_i$ and $\Gamma_i$ are obtained by solving problems (3.10) and (3.15) respectively, we have

$$\Gamma_i \leq \Delta_i.$$

It follows that

$$\bar{p}_i + \Gamma_i - \alpha_i - \max(-\bar{\alpha}, \Delta_i - \alpha_i) \leq \bar{p}_i + \Gamma_i - \Delta_i \leq \bar{p}_i.$$

In other words, the maximal magnitude of shock that a default bank $i$ can resist is bounded above by its total liability.

### 3.2. The indirect impact of the shock on other nodes in the system

In this subsection, we study the impact of a single shock to other nodes in the system. To start, we point out that Chen et al. (2016) and Glasserman and Young (2015) study the contagion impact of a single shock on other non-receiving nodes in the system. Different from the results in these two papers, we will estimate the magnitude of a single shock under which some non-receiving node will be bankrupted. For simplicity of discussion, throughout this subsection we assume that the first node is the receiving node with shock $s_1$. In such a case, we can consider the decomposed problem as follows

$$\begin{aligned} \max\ & x_j \\ s.t.\ & \bar{p}_1 x_1 - \sum_{j\neq 1} l_{j1} x_j \le \alpha_1 + s_1; \\ & \bar{p}_i x_i - \sum_{j\neq i} l_{ji} x_j \le \alpha_i,\ \forall i = 2, \cdots, n; \\ & x \le e. \end{aligned} \tag{3.19}$$

Particularly, we are mainly interested in estimating the minimum amount (denoted by $\bar{s}_{1j}$) of $s_1$ such that bank $j$ become bankrupted, i.e.,

$$\bar{s}_{1j} := \arg\max x_j^{(j)}(s_1) = 0, \tag{3.20}$$

where $x_j^{(j)}(s_1)$ denotes the objective function value at the optimal solution of problem (3.19). One can easily see that $\bar{s}_{11} = \max\{-\bar{\alpha}, \Delta_1 - \alpha_1\}$. Let us define

$$\bar{s}_1^{max} = \max_{j=1,\cdots,n.} \bar{s}_{1j}.$$

The following result follows directly from the above definition.

Proposition 3.1. *Suppose the system is triggered by a single shock $s_1$. If $s_1 < \bar{s}_1^{max}$, then some node in the system will be bankrupted.*

In what follows we consider the issue of how to estimate $\bar{s}_1^{max}$. Let us start by considering the issue of which node is more sustainable under the single shock $s_1$. For this, we introduce the following definition.

Definition 3.1. Let $\bar{s}_{1j}$ be defined by (3.20). We say a node $i$ is more sustainable than another node $j$ under the single shock $s_1$ if $\bar{s}_{1i} \le \bar{s}_{1j}$.

We next present a result on how to determine whether one node in the system is more sustainable than the receiving node itself . We have

Theorem 3.3. *Let $x^*(s_1)$ be the optimal solution of problem (3.10) with $i = 1$. For every $j > 1$, we have* *(i) If $x_j^*(s_1) > 0$, then node $j$ is more sustainable than the receiving node 1;*

*(ii) If $x_j^*(s_1) \leq 0$, then node $j$ is less sustainable than the receiving node;*

*(iii) If $x_j^*(s_1) > 0, \forall j > 1$, then $\bar{s}_1^{max} = \bar{s}_{11}$.*

*Proof.* Based on Conclusion (i) in Lemma 3.1, we can claim that $x_j^*(s_1)$ is nondecreasing in terms of $s_1$. By following a similar vein as in the proof of Proposition 2.1, we can show that

$$x_j^*(s_1) := \sup_{x(s_1) \in X(s_1)} x_j(s_1), \quad \forall j = 2, \cdots, n,$$

where $X(s_1)$ is the feasible set of problem (3.10). Since problem (3.10) and (3.19) have the same feasible set, $x^*(s_1)$ is a feasible solution of (3.19). Clearly, $x^*(s_1)$ maximizes $x_j$ and therefore, $x^*(s_1)$ is the unique optimal solution to (3.19). This implies that the optimal solution of problem (3.10) can be obtained by solving $n-1$ decomposed problem in the form of problem (3.19). Now, because

$$x_j^{(j)}(\bar{s}_{1j}) > 0, \quad \forall j > 1,$$

from (3.20) it follows

$$\bar{s}_{1j} \leq \bar{s}_{11}.$$

This proves the first conclusion in the theorem. The second conclusion follows similarly. We consider that $x_j^*(s_1) \leq 0$ which implies that

$$x_j^{(j)}(\bar{s}_{1j}) \leq 0, \quad \forall j > 1.$$

Based on (3.20) we have

$$\bar{s}_{1j} \geq \bar{s}_{11}.$$

This proves the second conclusion in the theorem. The last conclusion follows directly from the definition of $\bar{s}_1^{max}$. This completes the proof of the theorem. □

Theorem 3.3 provides a simple way to determine whether a non-receiving node in the system is more sustainable than the receiving node itself by solving problem (3.10). Note that when $\alpha > 0$, one can show that at the optimal solution $x^*(s_1)$ of problem (3.10), we have $x_j^* > 0$ for all $j > 1$. It follows from Theorem 3.3 that

Corollary 3.2. *Suppose a system is triggered by a single shock. If all the nodes in the system have positive assets, then every non-receiving node in the system is more sustainable than the receiving node itself.*

The above corollary shows that for problem (2.1), if the initial asset vector $\alpha$ is positive, and only $\alpha_1$ is subject to the shock $s_1$, then node 1 will be bankrupted first. Corollary 3.2 also implies that if the system is triggered by multiple shocks such that only one node receive a negative shock, then the node receiving the negative shock will become bankrupt first.

We next discuss how to estimate $\bar{s}_1^{max}$ when there exists some non-receiving node $j > 1$ that is less sustainable than the receiving node 1. For this, we recall conclusion (i) of Lemma 3.1, which shows that $x_j^*(s_1)$ is nondecreasing in terms of $s_1$. One way to locate $s_{1j}$ is applying a line search procedure based on the monotonicity of $x_j^*(s_1)$. Let us assume that problem (2.1) is feasible and thus, we can obtain an upper bound $u_s = 0$ for $s_{1j}$. From Theorem 2.1, we can also obtain a lower bound $l_s = -\bar{\alpha}$ for $s_{1j}$. We are now ready to describe a bisection search algorithm for locating $s_{1j}$.

**A Bisection Search Algorithm**

S.0 **Input:** $L$, $\bar{p}$, $\alpha$, $l_s = -\bar{\alpha}$, $u_s = 0$, and a stop criteria $\epsilon$;

S.1 **While** $u - l > \epsilon$ **Do**;

S.1.1 $s_1 := \frac{l_s + u_s}{2}$;

S.1.2 Solve problem (3.19);

S.1.3 **If** $x_j^*(s_1) > 0$, **then**

S.1.4 Set $u = s_1$,

S.1.5 **else**

S.1.6 Set $l = s_1$,

S.1.7 **endif**

S.2 **Output:** $s_{1j} = s_1$

For illustration, we use the same financial network as in Example 3.1 and adapt the bisection search algorithm to estimate $\bar{s}_{ij}$.

EXAMPLE 3.3. $L$ and $\alpha$ are the same as in example 3.1.

**Table 6 Bankruptcy Example in Financial Network**

| Optimal Solution | | | | |
|---|---|---|---|---|
| $x_1^*$ | $x_2^*$ | $x_3^*$ | $x_4^*$ | |
| 0.5329 | 1 | 0.7186 | 1 | $\alpha_1 = -9400$ |
| 0 | 1 | 0.4624 | 0.7454 | $\alpha_{1+} = \Delta_1 = -17894.736$ |
| -0.9551 | 1 | 0 | 0.2830 | $\alpha_{1-} = \alpha_1 - \bar{s}_{13} = -33017.514$ |
| 0 | 1 | 0.1856 | 0.2381 | $\alpha_{4-} = \alpha_4 - \bar{s}_{41} = -5187.8$ |

For the above example, we first estimate $\Delta_i, \forall i = 1, \cdots, 4$ by solving problem (3.10) which is also shown in Table 3. We also use the bisection search algorithm to estimate $\bar{s}_{ij}, \forall i, j = 1, \cdots, 4$, as listed in the following table.

**Table 7** $s_{ij}$ **estimated for Example 3.3**

| Node | $\bar{s}_{1j}$ | $\bar{s}_{2j}$ | $\bar{s}_{3j}$ | $\bar{s}_{4j}$ |
|---|---|---|---|---|
| 1 | -8494.736 | -27648.2 | -17546.0 | -17546 |
| 2 | -27648.2 | -27648.2 | -27648.2 | -27648.2 |
| 3 | -23617.514 | -27648.2 | -15881.652 | -23617 |
| 4 | -27648.2 | -27648.2 | -27648.2 | -22993.62 |

From Table 6, one can see that under a single shock $s_1$, the receiving node 1 is the least sustainable node and bankrupts first, while node 2 remains solvent as well as problem (2.1) is feasible. When the amount of the negative shock $|s_1|$ is sufficiently large such that $s_1 \leq \bar{s}_{13}$, then both nodes 1 node 3 become bankrupted.

From Table 7, we can see that if the single shock is received by either node 1 or 3, then the receiving node is the least sustainable node. However, when node 4 is the receiving node, then node 1 is the least sustainable node in the system. To verify this, we consider a negative shock $s_4 = \bar{s}_{41} = -17546$. As one can see from Table 6, node 1 becomes bankrupted under the shock $s_4 = -17546$ while node 4 survives.

## 4. The Vulnerability of A Financial Network under Multiple Shocks

In this section, we estimate the vulnerability of a financial network in a generic scenario where all the nodes are subject to market shocks. The section consists of three subsections. In the first subsection, we estimate the impact of asset inequality on the stability of the network, and show that, the least stable financial network can be attained at some network with a monopoly node. Then, we characterize conditions for bankruptcy in a financial network with a monopoly node. In the second and third subsections, we estimate the probabilities of insolvency and bankruptcy in the network respectively.

### 4.1. Asset inequality and stability of the financial system

In this subsection we estimate the vulnerability in a financial system when all the nodes are exposed to market shocks. First we point out that, as proved in Lemma 3.1, the solution $x^*(s)$ of problem (3.1) is component-wise monotone with respect to the market shock when only a single bank receives the shock. In such a case, we can conclude that $x^*(s)$ is also monotone in terms of total asset ($\bar{\alpha}$), i.e., the financial network will be more stable as the total asset increases. Such a result has also been used in Glasserman and Young (2015) to estimate the probability of insolvency in the network caused by the shock to a single node ($i$). It is of interests to see whether the monotone relationship between the repayments and the shocks still holds when all the nodes receive shocks. For this, we introduce the following definitions.

DEFINITION 4.1. A node $(i)$ in the financial system is said to be strictly solvent node if

$$x_i^* = 1, \quad [(\overline{P} - L^T)x^*]_i < \alpha_i$$

where, $x_i^*$ is the optimal solution of (2.1) or (2.2). Correspondingly the gap between the assets of one strictly solvent node and some default/bankrupted node in the system is called an asset inequality.

In this paper, we propose to measure the stability of a financial system in terms of the optimal objective value of problem (2.1). Based on this new measurement of stability, we can compare the stability of two financial systems with the same liabilities and different assets.

DEFINITION 4.2. Consider two financial systems with the same network structure $L$ with asset vectors $\alpha^1$ and $\alpha^2$ satisfying $e^T\alpha^1 = e^T\alpha^2 = \bar{\alpha}$. The first system is said to be less stable than the second one if

$$\bar{p}^T x^*(\alpha^1) < \bar{p}^T x^*(\alpha^2),$$

where $x^*(\alpha^1)$ and $x^*(\alpha^2)$ denote the optimal solution of (2.1) with $\alpha = \alpha^1$ and $\alpha = \alpha^2$, respectively. We remind the readers that the measurement of stability in our work is different from what used in the reference Acemoglu et al. (2013) where the authors suggested to use the number of nodes affected by the shock to measure the stability of the system.

Now let consider a financial network with total asset $\bar{\alpha}^1$ where problem (2.1) is feasible and there exists some default bank, and strictly solvent node. Without loss of generality, let us further assume that bank 1 defaults, i.e., at the optimal solution $x^*$ of problem (2.2), we have $0 < x_1^* < 1$, and bank $n$ is strictly solvent, i.e., $x_n^* = 1$, $[(\overline{P} - L^T)x^*]_n < \alpha_n$. Now let us consider a new financial system with total asset $\bar{\alpha}^2 = \bar{\alpha}^1$ such that $\alpha_1^2 = \alpha_1^1 - \epsilon$ and $\alpha_n^2 = \alpha_n^1 + \epsilon$. Since $x_n^*(\alpha^1) = 1$, it holds $x_n^*(\alpha^2) = 1$. It follows from Lemma 3.1 that

$$x_1^*(\alpha^2) < x_1^*(\alpha^1), \quad x_i^*(\alpha^2) \le x_i^*(\alpha^1), \forall i = 2, \cdots, n.$$

From the above discussion we immediately obtain the following result.

PROPOSITION 4.1. *Suppose that the summation of the assets of one default node and another strictly solvent node in a financial network remain invariant. A larger asset inequality between these two nodes will decrease the stability of the network.*

We remark that since increasing the asset value $\alpha_n$ will not help to improve the stability of the underlying network, this implies that an increase in the total asset may not improve the stability of the network. The following example demonstrates such an phenomenon.

Example 4.1. We consider the same data matrix as in Example 2.1 with three different vectors of asset

$$\alpha^1 = (-400, 3790, 100, 1358.2)^T, \alpha^2 = (-500, 4000, 100, 1358.2)^T,$$

$$\alpha^3 = (-1000, 4000, 100, 3358.2)^T$$

such that at least one of the nodes in the system defaults.

**Table 8 Stability of a Financial System VS Asset Inequality.**

| Optimal Solution | | | | |
|---|---|---|---|---|
| $x_1^*$ | $x_2^*$ | $x_3^*$ | $x_4^*$ | |
| 0.9842 | 1 | 0.9987 | 1 | $\alpha^1$ |
| 0.980 | 1 | 0.9978 | 1 | $\alpha^2$ |
| 0.959 | 1 | 0.9932 | 1 | $\alpha^3$ |

In the above example, we change the value of $\alpha^1$ such that the asset value of the first node is decreased while the asset values of all other nodes are increased and thus, the total asset is increased as well. As you can see from Table 8, we have

$$x_1^*(\alpha^1) > x_2^*(\alpha^2) > x_3^*(\alpha^3), \quad \bar{\alpha}^1 < \bar{\alpha}^2 < \bar{\alpha}^3.$$

The example shows clearly that an increase in the total asset may not improve the stability of the financial network. With a close look at the example, we find that the asset inequality has also increased as the total asset grows. This illustrates that the asset inequality in the financial network has a negative effect on the stability of the network. Next we study the stability of a network with a dominant node, which represents an extreme distribution of the assets defined as follows.

Definition 4.3. A node $i$ in a financial network is said to be a monopoly node if

$$\alpha_i = \bar{\alpha}, \quad \alpha_j = 0, \quad \forall j = 1, \cdots, i-1, i+1, \cdots, n, \tag{4.1}$$

We next show that under the assumption that total asset is fixed, the network is the least stable one when it has a monopoly node. To start, let us consider the following problem

$$\begin{aligned} \min_{\alpha} \max_{x} \quad & \bar{p}^T x \\ s.t. \quad & (\bar{P} - L^T)x \leq \alpha; \\ & x \leq e; \\ & \sum_i \alpha_i = \bar{\alpha}; \\ & \alpha \geq 0. \end{aligned} \tag{4.2}$$

For a given $\alpha$, let $f(\alpha)$ denote the objective function value of problem (2.2). Then we can rewrite the above problem as the following

$$\begin{aligned}\min_{\alpha}\ & f(\alpha) \\ s.t.\ & \sum_i \alpha_i = \bar{\alpha}; \\ & \alpha \geq 0.\end{aligned} \tag{4.3}$$

Note that by using the duality theorem for linear optimization, we have

$$\begin{aligned} f(\alpha) := \min_{\lambda} \quad & \alpha^T \lambda + e^T \bar{p} - e^T(\overline{P} - L)\lambda \\ s.t. \quad & (\overline{P} - L)\lambda \leq \bar{p} \\ & \lambda \geq 0 \end{aligned}$$

From the above definition, one can see that $f(\alpha)$ is concave with respect to $\alpha$. Therefore, the optimal solution of (4.3) can be obtained in one of the extreme points of the constrained set, which is precisely a network with a monopoly node. From this, we immediately have the following result.

Proposition 4.2. *Suppose that the total asset is fixed. The least stable financial network can be attained at some network with a monopoly node. Moreover, if the monopoly node in a network is strictly solvent and there exists some default node in the system, then the stability of network can be improved by redistributing the assets in the network.*

*Proof.* We need only to prove the second conclusion of the proposition. Let assume that the first node is the monopoly node that is strictly solvent. Thus, we have

$$x_1^{(1)} = 1, \quad [(\overline{P} - L^T)x^{(1)}]_1 < \alpha_1.$$

In this case, there exist at least one default node $i > 1$ in the system such that

$$x_i^{(1)} < 1.$$

Now if we slightly increase $\alpha_i^1$ to $\alpha_i^2 = \alpha_i^1 + \epsilon$ and decrease $\alpha_1^1$ to $\alpha_1^2 = \alpha_1^1 - \epsilon$ such that $[(\overline{P} - L^T)x^{(1)}]_1 \leq \alpha_1^1 - \epsilon$, we can conclude that

$$x^{(2)} > x^{(1)},$$

where $x^{(1)}$ and $x^{(2)}$ are the optimal solutions of (2.1) when $\alpha = \alpha^1$ and $\alpha = \alpha^2$ respectively. Note that the above inequality follows from conclusion (ii) in Lemma 3.1 which states that $x_i^{(1)}(s_i)$ is strictly increasing in terms of $s_i$. This finishes the proof of the proposition. □

Proposition 4.2 shows that the worst-case scenario for problem (2.1) w.r.t. the asset distribution is the network with a monopoly node.

### 4.2. Probability analysis on the vulnerability of a financial network

In this subsection we explore the vulnerability of a network with a monopoly node when all the nodes receive shocks under certain assumption of the shock distribution. First, we recall that, as illustrated by Example 4.1, the monotonicity between the repayments ($x^*(s)$) and the shocks ($s$) does not hold in general if all the nodes receive shocks. Fortunately, as shown in Conclusion (ii) of Theorem 2.2, a node $i$ is insolvent if $\alpha_i \leq \bar{p}_i - (L^T e)_i$. Since the total liability of a financial institution $\bar{p}_i$ and its total claims $(L^T e)_i$ are usually known in advance, this allows us to estimate the probability of insolvency in the financial network under the following assumption.

Assumption 4.1. *(i) The financial network is well-balanced and the assets are nonnegative;*

*(ii) All the shocks follow the same independent normal distribution with a zero mean and variance $\sigma^2$, i.e., $s_i \sim \mathcal{N}(0, \sigma^2)$.*

We have

Theorem 4.1. *Suppose that Assumption 4.1 holds. For a fixed total asset ($\bar{\alpha}$), the following conclusions hold.*

*(i) The network with a monopoly node has the highest probability of insolvency and is the most vulnerable one. Moreover, it holds*

$$P(\exists i : x_i^* < 1) \geq 1 - (0.5)^{n-1}. \tag{4.4}$$

*(ii) The system is most stable when the assets are evenly distributed, i.e, $\alpha_i = \frac{\bar{\alpha}}{n}, \forall i = 1, \cdots, n$.*

*Proof.* We start with the first conclusion. Under Assumption 4.1, from Conclusion (ii) of Theorem 2.2, we obtain that $x_i^* < 1$, if $\alpha_i < \bar{p}_i - (L^T e)_i = 0$. Thus, we have

$$P(\exists i : x_i^* < 1) = P(\exists i : \alpha_i + s_i < 0).$$

Therefore, the most vulnerable network can be identified via solving the following optimization problem

$$\begin{aligned} \min_{\alpha} \quad & \prod_i (1 - P(s_i < -\alpha_i)) = \prod_i F(\alpha_i) \\ s.t. \quad & \sum_i \alpha_i = \bar{\alpha}; \\ & \alpha_i \geq 0, \end{aligned} \tag{4.5}$$

where $F(\cdot)$ is the cumulative distribution function of normal distribution with a density function $f(\cdot)$. Note that the equality in the objective function follows from the symmetry of the shock distribution. Since $F(\cdot)$ is strictly monotone, we can rewrite the problem as follows

$$\min_{\alpha} \quad \sum_i \ln F(\alpha_i) \tag{4.6}$$

$$s.t. \quad \sum_i \alpha_i = \bar{\alpha};$$
$$\alpha_i \geq 0,$$

Because the objective and constraint functions in problem (4.6) are differentiable, the optimal solution to the above problem must satisfy the Karush-Kuhn-Tucker (KKT) conditions (Boyd and Vandenberghe 2004), i.e.,

$$\frac{\partial \ln F(\alpha_i)}{\partial \alpha_i} + \lambda - \nu_i = 0, \quad \forall i = 1, \cdots, n; \tag{4.7}$$
$$\sum_i \alpha_i = \bar{\alpha} \tag{4.8}$$
$$\nu_i \alpha_i = 0, \quad \forall i = 1, \cdots, n; \tag{4.9}$$
$$\alpha_i \geq 0, \quad \forall i = 1, \cdots, n;$$
$$\nu_i \geq 0, \quad \forall i = 1, \cdots, n.$$

where $\lambda$ and $\nu$ are the Lagrange multipliers. From (4.9), it is easy to see that for every index $i$, either $\alpha_i$ or $\nu_i$ must be zero. Let us define the index sets based on the element values of $\nu$ by

$$I_0 = \{i : \nu_i = 0\}, \quad I_1 = \{i : \nu_i \neq 0\}. \tag{4.10}$$

It follows immediately that

$$\alpha_i = 0, \quad \forall i \in I_1.$$

Using the above relation, we can rewrite the KKT conditions as

$$\frac{f(\alpha_i)}{F(\alpha_i)} + \lambda = 0, \ \forall i \in I_0;$$
$$\alpha_i = 0, \ \forall i \in I_1;$$
$$\sum_{i \in I_0} \alpha_i = \bar{\alpha}.$$

Since the function $f(\cdot)/F(\cdot)$ is a bijective map, from the above relations we can claim that

$$\alpha_i = \frac{\bar{\alpha}}{k}, \quad \forall i \in I_0,$$

where $k$ is the number of nodes in $I_0$. Based on this, the objective function of (4.5) can be written as a function of $k$, i.e.,

$$g(k) = F(\bar{\alpha}/k)^k F(0)^{(n-k)} = (2F(\bar{\alpha}/k))^k (1/2)^n.$$

We next show that $g(k)$ is an increasing function with respect to $k$. For this, we take the first derivative of $\ln g(k)$ with respect to $k$ as follows.

$$\partial \ln g(k)/\partial k = \ln 2F(\bar{\alpha}/k) - (\bar{\alpha}/k) f(\bar{\alpha}/k)/F(\bar{\alpha}/k). \tag{4.11}$$

We also have that

$$\lim_{k\to\infty} \ln 2F(\bar{\alpha}/k) - (\bar{\alpha}/k)f(\bar{\alpha}/k)/F(\bar{\alpha}/k) = 0. \tag{4.12}$$

Now, if we take the second derivative of $\ln g(k)$ with respect to $k$, we have

$$\partial^2 \ln g(k)/\partial k^2 \;= -\bar{\alpha}^3 f(\bar{\alpha}/k)/\sigma^2 F(\bar{\alpha}/k) - \bar{\alpha}^2 f(\bar{\alpha}/k)^2/k^3 F(\bar{\alpha}/k)^2 < 0. \tag{4.13}$$

From (4.12) and (4.13), we can conclude that for small value of $k$, $\frac{\partial \ln g(k)}{\partial k} > 0$, which implies that $g(k)$ is increasing in $k$. Based on this, one can conclude that the objective value of (4.5) attains its minimum when $k = 1$, i.e., there exists an index $i$ such that

$$\alpha_j^* = \bar{\alpha}, \quad \alpha_i^* = 0, \quad \forall i \neq j.$$

From the above relation and the symmetry of the shock distribution we obtain

$$\begin{aligned} P(\exists i : x_i^* < 1) \; &= 1 - \prod_i (1 - P(s_i < -\alpha_i)); \\ &\geq 1 - \prod_{i \neq j} (1 - P(s_i < -\alpha_i)); \\ &= 1 - (0.5)^{n-1}, \end{aligned}$$

where the inequlity follows from the fact that $(1 - P(s_j < -\alpha_j)) \leq 1$.

Similarly, since $k \leq n$, we can conclude that the maximum value of $g(k)$ can be obtained when the assets are evenly distributed as below.

$$\alpha_i^* = \frac{\bar{\alpha}}{n}, \qquad \forall i = 1, \cdots, n.$$

This finishes the proof of the theorem. □

Theorem 4.1 shows that for problem (2.1), the monopoly case is the most vulnerable in terms of probability of insolvency. We note that Glasserman and Young (2015) estimated the probability of a subset of nodes to default caused by the shock received by a single node not in that subset, and obtained a similar result as Conclusion (ii) of Theorem 4.1 (see Proposition 1 and Corollary 1 in Glasserman and Young (2015)). Chen et al. (2016) also extend the results in Glasserman and Young (2015) by considering the impact of the liquidity concentration on the bank. It is worthwhile mentioning that the proof of the theorem depends only on the two properties that the function $(f(\cdot)/F(\cdot))$ is bijective, and the function $g(k)$ is decreasing. This implies that the results in Theorem 4.1 can be extended to any shocks that follows some distribution with a density function and an accumulative function such that these two properties are satisfied.

### 4.3. Probability of bankruptcy in the system

In this subsection, we estimate the probability of bankruptcy in the system. Throughout this subsection, we make the following assumption.

ASSUMPTION 4.2. *All the shocks follow some independent normal distributions with a zero mean and variance $\sigma_i^2$, i.e., $s_i \sim \mathcal{N}(0,\sigma_i^2)$.*

THEOREM 4.2. *Suppose that Assumption 4.2 holds. Then the probability that some bank ($i = 1,\cdots,n$) will be bankrupted is larger than*

$$1 - \prod_i (1 - P(s_i \le \Delta_i - \alpha_i)). \tag{4.14}$$

*Proof.* Without loss of generality, we assume $x_i^* = 0$. Given any $i$, we rewrite the reduced problem (3.10) w.r.t. the shock $s$ as follows.

$$\begin{aligned} -\Delta_i(s) = \max\ & \sum_{j\neq i} l_{ji} x_j \\ s.t.\ & \bar{p}_j x_j - \sum_{k\neq j, k\neq i} l_{kj} x_k \le \alpha_j + s_j,\ \forall j \neq i; \\ & x_j \le 1,\ \forall j \neq i. \end{aligned} \tag{4.15}$$

The dual problem of (4.15) reads as

$$\begin{aligned} -\Delta_i(s) = \min\ & \sum_{j\neq i} (\alpha_j + s_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j + \sum_{j\neq i} l_{ji} \\ s.t.\ & \bar{p}_j y_j - \sum_{k\neq j, k\neq i} l_{jk} y_k \le l_{ji},\ \forall j \neq i; \\ & y \ge 0. \end{aligned} \tag{4.16}$$

Let $\mathcal{Y}$ denotes the feasible set of problem (4.16). Since all the variables $s_j$ $\forall j = 1,\cdots,n$ are independent random variables $s_j \sim \mathcal{N}(0,\sigma_j^2)$, it follows from Theorem 3.1 that

$$P(x_i^* \le 0) \ = P(s_i \le -\mathbf{E}[\min_{y\in\mathcal{Y}} \sum_{j\neq i} (\alpha_j + s_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j^i + \sum_{j\neq i} l_{ji}] - \alpha_i),$$

where $\mathbf{E}[\cdot]$ denotes the expected value of $-\Delta_i(s)$. Note that the dual problem (4.16) is a linear optimization problem parameterized by the random noise $s$. From a computational perspective, we can use stochastic programming and scenario generation to estimate the probability $P(x_i^* < 0)$. In what follows we present a simple way to obtain a lower bound on the the probability $P(x_i^* < 0)$. Since the objective function in problem (4.16) is linear in terms of $s$, $-\Delta_i(s)$ is concave with respect to $s$. It follows immediately

$$\mathbf{E}[\min_{y\in\mathcal{Y}} \sum_{j\neq i} (\alpha_j + s_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j + \sum_{j\neq i} l_{ji}]$$

$$
\begin{aligned}
&\ge \min_{y\in\mathcal{Y}} \sum_{j\neq i} \mathbf{E}(\alpha_j + s_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j + \sum_{j\neq i} l_{ji} \\
&= \min_{y\in\mathcal{Y}} \sum_{j\neq i} \mathbf{E}(\alpha_j - \bar{p}_j + \sum_{k\neq j, k\neq i} l_{jk}) y_j + \sum_{j\neq i} l_{ji} \\
&= -\Delta_i,
\end{aligned}
$$

where the first inequality follows from the concavity of $-\Delta_i(s)$ w.r.t. $s$ and the fact that all the parameters in (4.16) except $s$ are constant, and the first equality follows from the assumption $\mathbf{E}[s_j] = 0,\ \forall j = 1, \cdots, n$. Thus, we have

$$
P(x_i^* \le 0) \ge P(s_i \le \Delta_i - \alpha_i), \tag{4.17}
$$

which further implies

$$
1 - \prod_i (1 - P(x_i^* \le 0)) \ge 1 - \prod_i (1 - P(s_i \le \Delta_i - \alpha_i)).
$$

This completes the proof of the theorem. □

Theorem 4.2 provides an estimation on the lower bound for the probability that some bank will be bankrupted in the system. Such an estimate depends on the current asset vector and interbank liability matrix. Therefore, it may help the clearing agent to identify the future risk of bankruptcy in the system based on the current market information, and implement policies to avoid such a risk in advance.

We next study the vulnerability of a specific network with monopoly node under some assumption about the monopoly node in it. For this, we first introduce the following definition.

Definition 4.4. A node $i$ in the financial network is said to be liability-free, if the following relation holds.

$$
l_{ij} = 0, \quad \forall j = 1, \cdots, n, j \neq i.
$$

We have

Theorem 4.3. *Given a network with a monopoly node in which the monopoly node is liability-free, and all the other nodes are well-balanced. Suppose that Assumption 4.2 hold. Then we have*

$$
P(\exists j : x_j^* \le 0) \ge 0.5.
$$

*Proof.* W.l.o.g., we assume that the first node is the monopoly node. In this case, we have $l_{1j} = 0, \forall j = 1, \cdots, n$, and $\alpha_i = 0, \forall i \neq 1$. In such a case, we can rewrite problem (2.2) as follows

$$
\begin{aligned}
\max \quad & \sum_i \bar{p}_i x_i && (4.18)\\
s.t. \quad & \bar{p}_1 x_1 - \sum_{k\ge 2} l_{k1} x_k \le \alpha_1; \\
& \bar{p}_j x_j - \sum_{k\ge 2} l_{kj} x_k \le s_j,\ \forall j = 2, \cdots, n; && (4.19)\\
& x_j \le 1,\ \forall j = 1, \cdots, n.
\end{aligned}
$$

By summing up the left-had sides of the constraints (4.19), we obtain

$$\sum_{j=2}^{n}\left(\bar{p}_j x_j - \sum_{k\geq 2} l_{kj}x_k\right) = \sum_{k\geq 2} l_{k1}x_k \leq \sum_{j=2}^{n} s_j, \tag{4.20}$$

where the first equality follows from the fact that all the non-monopoly nodes are balanced. From (4.20), we can conclude that if all the elements of $x$ are positive, then it must hold $\sum_{j=2}^{n} s_j > 0$. In other words, if $\sum_{j=2}^{n} s_j \leq 0$, then $x$ must have some element less than or equal 0. It follows from the symmetry of the shock distribution that

$$P(\exists j : x_j^* \leq 0) \geq P(\sum_{i=1}^{n} s_i \leq 0) = 0.5.$$

This completes the proof of the theorem. □

Theorem 4.3 shows the high risk in a financial network with a monopoly node that is liability-free. The theorem also implies that if a monopoly node in the system bankrupts, then there is a high probability of bankruptcy in the reduced system consisting of all the non-monopoly nodes.

We next provide some numerical examples to verify our theoretical results.

EXAMPLE 4.2. We consider the same data matrix as in Example 2.1. The vector of asset $\alpha^1 = (7674.5, 7674.5, 7674.5, 7674.5)^T$ is computed based on data from Table 3 in Chen et al. (2016) ( In a recent FDIC report FDIC Bank Data and Statistics (2013), the total equity (which we call asset in this paper) is reported to be about 14.4% of the total liability.). We also generate a new asset vector $\alpha^2 = (30698, 0, 0, 0)^T$ corresponding to a system with a monopoly node.

**Table 9 Bankruptcy in a Financial Network with Random Shock**

| $0 < x_i^* < 1$ | $x_i^* \leq 0$ | $\sum_i(\alpha_i + s_i) < 0$ | |
|---|---|---|---|
| 9493 | 2 | 505 | $\alpha^1$ |
| 8854 | 582 | 564 | $\alpha^2$ |

| Node | $\alpha^1$ | | $\alpha^2$ | |
|---|---|---|---|---|
| | $\Delta_i$ | $P(s_i \leq \Delta_i - \alpha_i)$ | $\Delta_i$ | $P(s_i \leq \Delta_i - \alpha_i)$ |
| 1 | -23023.5 | 0.0000013 | 0.000028 | 0.0000013 |
| 2 | -23023.5 | 0.0000040 | -15833.2 | 0.0107076 |
| 3 | -23023.5 | 0.0033758 | -21063.5 | 0.0315362 |
| 4 | -23023.5 | 0.0053317 | -21873.7 | 0.0344167 |

We use MATLAB function NORMRAND to generate random vectors of shock with Normal distribution under the assumption that $s_i \sim \mathcal{N}(0, \sigma_i^2),\ \forall i = 1, \cdots, n$. For each node $i$, we set $\sigma_i = 0.25\bar{p}_i$. The results are summarized in Table 9.

Since $\alpha^1 > 0$ and the system is well-balanced, all the nodes in the system are solvent when there is no market shocks. As shown in Table 9, there are 507 bankruptcy cases observed in our experiments under random shocks, showing an empirical probability bound $P(x_i^* < 0) = 0.0507$. For each $i = 1, 2, 3, 4$, we also compute the value of $\Delta_i$ via solving problem (3.10). Based on the values of $\Delta_i$s, we can obtain the following bound on probability of bankruptcy in the system

$$1 - \prod_i (1 - P(s_i \le \Delta_i - \alpha_i)) = 0.0086 \le 0.0507.$$

We next consider a system with a monopoly node (node 1) and the asset vector $\alpha^2$. Note that all the banks in the system are solvent if there is no shocks in the market. According to Theorem 4.1, it has the highest probability of insolvency. As shown in Table 9, the empirical probability of bankruptcy is 0.1146. Based on the computed values of $\Delta_i$s, we obtain a lower bound as follows

$$1 - \prod_i (1 - P(s_i \le \Delta_i - \alpha_i)) = 0.0748 \le 0.1146.$$

One can easily see from Table 9 that the values of $\Delta_i$s are much larger in a network with a monopoly node. One can also see from Table 9 that for all the randomly generated scenarios of the system with a monopoly node, there are insolvent nodes in the system, indicating an empirical probability 1 of insolvency. This is consistent with Theorem 4.1, which provides a lower bound for the probability of insolvency as below:

$$P(\exists i : x_i^* < 1) \ge 1 - (0.5)^3 = 0.875.$$

We next consider a financial network with a liability-free monopoly node. The liability matrix and the asset vector are given in the following table.

EXAMPLE 4.3. **Table 10 Bankruptcy in a financial network with liability free monopoly node**

| Liability Matrix | | | | | | |
|---|---|---|---|---|---|---|
| Node | 1 | 2 | 3 | 4 | $\bar{p}$ | $\alpha$ |
| 1 | 0 | 0 | 0 | 0 | 0 | 30698 |
| 2 | 4857 | 0 | 10625 | 12047 | 27529 | 0 |
| 3 | 9971 | 10625 | 0 | 24734 | 45330 | 0 |
| 4 | 11306 | 12047 | 24734 | 0 | 48087 | 0 |
| Claims | 26134 | 22672 | 35359 | 367810 | 120946 | 30698 |

| $0 < x_i^* < 1$ | $x_i^* \le 0$ | $\sum_i(\alpha_i + s_i) < 0$ |
|---|---|---|
| 4057 | 5518 | 371 |

We use the same procedure as in Example 4.2 to generate random shocks. As shown in Table 10, there are 5889 bankruptcy cases observed, showing an empirical probability of bankruptcy $P(x_i^* < 0) = 0.5889$, which is larger than the theoretical bound provided by Theorem 4.3.

## 5. The Domino Effect of Bankruptcy in A Financial Network with A Monopoly Node

In this section, we estimate the domino effect of bankruptcy in a well-balanced network with a monopoly node. This is motivated by several observations. First of all, as shown in the previous section, the network with a monopoly node is the most vulnerable one in terms of asset distribution. Second, massive bankruptcies had been observed during the 2007-2008 financial crisis. Due to the severe consequence of the domino effect of bankruptcy, it is of interests to explore the network structure under which the bankruptcy of some node in the network will cause other nodes to be bankrupted.

The following result regarding the domino effect of the monopoly node's bankruptcy is intuitive. For self-completeness, we also include a proof.

PROPOSITION 5.1. *Given a fully connected and well-balanced financial network with a monopoly node. If the monopoly node is bankrupted, then all other nodes in the system will be bankrupted as well.*

*Proof.* For simplicity we assume that the first node is the monopoly node. Therefore, we can rewrite problem (2.2) as follows

$$\begin{aligned} \max \quad & \bar{p}^T x && (5.1) \\ s.t. \quad & \bar{p}_1 x_1 - \sum_{k>2} l_{k1} x_k \leq \bar{\alpha}; \\ & \bar{p}_j x_j - \sum_{k>2} l_{kj} x_k - l_{1j} x_1 \leq 0, \ \forall j = 2, \cdots, n-1; && (5.2) \\ & x \leq e. \end{aligned}$$

Now let us consider that the monopoly node is bankrupted, i.e., $x_1^* \leq 0$. In this case, we first show that at the optimal solution of (5.1) we have

$$x_j^* = c, \quad \forall j \geq 2, \quad c \in \Re. \tag{5.3}$$

To show this, suppose to the contrary that (5.3) does not hold. Then there exists an index $j^*$ such that $x_{j*}^* = \max_{j=2,\cdots,n} x_j^*$. Based on this, by rewriting feasibility condition (5.2) for index $j^*$, we have

$$\bar{p}_{j^*} x_{j^*}^* - \sum_{k>2} l_{kj^*} x_k^* \leq l_{1j^*} x_1^* \leq 0.$$

Using the connectivity of the underlying network and following a similar vein as in the proof of Lemma 2.1, we can conclude that (5.3) holds at the optimal solution of problem (5.1). Now it suffices to show that $c \leq 0$. Since the network is fully connected, there exists $m \geq 2$ satisfying $l_{1m} > 0$. Now recall the feasibility condition (5.2) for index $m$, we have

$$l_{1m}x_m^* = (\bar{p}_m - \sum_{k>2} l_{km})x_m^* \leq 0,$$

where the equality follows from the fact that the network is well-balanced. This implies that $x_m^* = c \leq 0$, which completes the proof of the proposition. $\square$

Proposition 5.1 illustrates that the bankruptcy of the monopoly node will cause all other nodes in the system to be bankrupted, which is consistent with the "too big to fail" theory. Next we study how the bankruptcy of a non-monopoly will affect the system. For this, we consider a well-balanced network with a tridiagonal structure specified below.

Assumption 5.1. *A financial network is said to have a tridiagonal structure if it satisfies the following relations:*

$$l_{ij} > 0, \quad \forall (i,j) \in \{(i,j) : |i-j| \leq 1, \ \forall i,j = 1, \cdots, n\}. \tag{5.4}$$

We remark that the above structure is identified in recent work by Aein and Peng (2017). Next we study the domino effect of bankruptcy in a balanced network with a tridiagonal structure and the first node as a monopoly node.

Proposition 5.2. *Given a well-balanced network with a tridiagonal structure and a monopoly node. If a non-monopoly node is bankrupted, then all the nodes following it will be bankrupted.*

*Proof.* Let us rewrite problem (2.2) as follows

$$\begin{aligned} \max \quad & \bar{p}^T x && (5.5) \\ s.t. \quad & \bar{p}_1 x_1 - l_{21} x_2 \leq \bar{\alpha}; \\ & \bar{p}_j x_j - l_{(j-1)j} x_{j-1} - l_{(j+1)j} x_{j+1} \leq 0, \ \forall j = 2, \cdots, n-1; && (5.6) \\ & \bar{p}_n x_n - l_{(n-1)n} x_{n-1} \leq 0; \\ & x \leq e. \end{aligned}$$

Since the network is well-balanced and has a tridiagonal structure, we have $\bar{p}_n = l_{(n-1)n}$ and $\bar{p}_1 = l_{21}$. Based on this, at the optimal solution of (5.5), we have

$$x_n^* \leq x_{n-1}^*. \tag{5.7}$$

Now, let us consider (5.6) for $j = n-1$. In this case, we have,

$$\begin{aligned} & \bar{p}_{n-1} x_{n-1} - l_{(n-2)(n-1)} x_{n-2} - l_{(n)(n-1)} x_n \\ & = l_{(n-2)(n-1)}(x_{n-1} - x_{n-2}) + l_{(n)(n-1)}(x_{n-1} - x_n) \leq 0 \qquad (5.8) \end{aligned}$$

The equality follows from the fact that $\bar{p}_{n-1} = l_{(n-2)(n-1)} + l_{(n)(n-1)}$. From (5.7) and (5.8), we have

$$x_n^* \leq x_{n-1}^* \leq x_{n-2}^*.$$

Following a similar procedure for $j \leq n-2$, we can obtain

$$x_n^* \leq x_{n-1}^* \leq \cdots \leq x_1^*.$$

This implies that when a non-monopoly node $i$ become bankrupted, i.e., $x_i^* \leq 0$, we have

$$x_n^* \leq x_{n-1}^* \leq \cdots \leq x_i^* \leq 0.$$

This finishes the proof of this proposition. □

Proposition 5.2 shows that if the network has a tridiagonal structure and a solvent monopoly node, then the bankruptcy of every non-monopoly node will still have a significant domino effect. In other words, the solvency of big banks cannot avoid massive bankruptcies if the financial network has a tridiagonal structure and is dominated only by a few big banks. We remark that during the 2007-2008 financial crisis, the federal government bailed out numerous major banks to stabilize the market. Nevertheless, a large number of banks still bankrupted during the crisis period. Proposition 5.2 provides an interesting interpretation to such a phenomenon.

Next, we provide a numerical example to verify the conclusions in the proposition.

Example 5.1. We consider a tridiagonal financial network with four banks in which the first node is the monopoly node ($\bar{\alpha} = \alpha_1$). The liability matrix is extracted from the liability matrix (see Table 8 in Chen et al. (2016)) by considering the first four banks in the network. The asset vector and the optimal solutions are also listed below.
In this example, a shock of magnitude $s_1 < -\bar{\alpha}$ will cause node 1 to be bankrupted ($x_1^* < 0$). Following this, the whole system will be bankrupted. In other words, the bankruptcy of the monopoly node will be propagated to the whole network. This domino phenomenon is consistent with the so-called "too big to fail" theory, which advocates for government's intervenience in a period of financial crisis.

Example 5.1 demonstrates that not only the failure of the monopoly node in the network, but also the failure of a non-monopoly node may lead to a catastrophic disaster. As one can see from Table 11, under a negative shock of magnitude $s_2 = -31855$ triggering node 2 we have $x_2^* = 0$. Following this, we have $x_3^* = x_4^* = 0$. This shows that the monopoly network with a tridiagonal structure is very fragile, and the bankruptcy of a non-monopoly node in the network may have a domino effect too.

**Table 11 Domino Effect of Bankruptcy in a Tridiagonal Financial Network with a Monopoly Node.**

| Liability Matrix | | | | | | |
|---|---|---|---|---|---|---|
| Node | 1 | 2 | 3 | 4 | $\bar{p}$ | $\alpha$ |
| 1 | 0 | 31855 | 0 | 0 | 31855 | 32698 |
| 2 | 31855 | 0 | 18016 | 0 | 49871 | 0 |
| 3 | 0 | 18016 | 0 | 73185 | 91201 | 0 |
| 4 | 0 | 0 | 73185 | 0 | 73185 | 0 |
| Claims | 31855 | 49871 | 91201 | 73185 | 246112 | 32698 |

| Optimal Solution | | | | |
|---|---|---|---|---|
| $x_1^*$ | $x_2^*$ | $x_3^*$ | $x_4^*$ | |
| 1 | 1 | 1 | 1 | $\alpha$ |
| 1 | 0 | 0 | 0 | $\alpha_{2^-} = -31855$ |

## 6. Conclusions

In this paper, we study the vulnerability of the financial network via analyzing the infeasibility of Eisenberg-Noe's linear optimization model and its relaxation. We show that as long as the total asset is nonnegative, the relaxation model is feasible. Under the assumption that only a single bank is exposed to market shock, we characterize conditions under which a single bank is solvent, default, or bankrupted.

For the generic scenario where all banks are triggered by market shocks, we show that both the total asset and the asset inequality may affect the stability of financial network. Particularly, we show that while a larger total asset may not improve the stability of the network, a larger asset inequality will reduce the stability of the network. We estimate the probability of insolvency and the probability of bankruptcy under certain assumptions on network structure and shock distribution. Particularly, we carry out a deterministic analysis showing that the least stable network can be attained at some network with a monopoly node, and show that the such a network has the highest probability of insolvency and is the most vulnerable network. We also study the contagious effect of bankruptcy under the network with a monopoly node and tridiagonal structure.

Several issues are of interests for future research. The first issue is how to identify the structure of the liability matrix such that the resulting system is the most or least stable one. Progress in such a topic will provide insights for the clearing agent on which kind of policies should be implemented in advance to prevent a catastrophic disaster. The second issue is, though we have provided a lower bound for the probability of bankruptcy in the network, we do observe in our experiments that there is a large gap between the empirical bound and the theoretical lower bound. Further study is needed to close such a gap. Finally we point out that after the financial crisis in 2007-2008, new regulations have been implemented/enforced for the financial market. It will be interesting to incorporate the new regulations in the assessment of systemic risk.

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